An analytical approximation scheme to two-point boundary value problems of ordinary differential equations

2007 J. Phys. A: Math. Theor. 40 F215

(http://iopscience.iop.org/1751-8121/40/9/F01)

IOP FTC

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) F215–F221 doi:10.1088/1751-8113/40/9/F01

FAST TRACK COMMUNICATION

An analytical approximation scheme to two-point boundary value problems of ordinary differential equations

Bruno Boisseau[1], **Péter Forgács**[1,2] **and Hector Giacomini**[1]

[1] Laboratoire de Mathématiques et Physique Théorique CNRS, Université François Rabelais Tours, Fédération Denis Poisson Parc de Grandmont, 37200 Tours, France

[2] MTA RMKI, H-1525 Budapest, PO Box 49, Hungary

Received 5 December 2006
Published 14 February 2007
Online at stacks.iop.org/JPhysA/40/F215

Abstract

A new (algebraic) approximation scheme to find *global* solutions of two-point boundary value problems of ordinary differential equations (ODEs) is presented. The method is applicable for both linear and nonlinear (coupled) ODEs whose solutions are analytic near one of the boundary points. It is based on replacing the original ODEs by a sequence of auxiliary first-order polynomial ODEs with constant coefficients. The coefficients in the auxiliary ODEs are uniquely determined from the local behaviour of the solution in the neighbourhood of one of the boundary points. The problem of obtaining the parameters of the global (connecting) solutions, analytic at one of the boundary points, reduces to find the appropriate zeros of algebraic equations. The power of the method is illustrated by computing the approximate values of the 'connecting parameters' for a number of nonlinear ODEs arising in various problems in field theory. We treat in particular the static and rotationally symmetric global vortex, the skyrmion, the Abrikosov–Nielsen–Olesen vortex, as well as the 't Hooft–Polyakov magnetic monopole. The total energy of the skyrmion and of the monopole is also computed by the new method. We also consider some ODEs coming from the exact renormalization group. The ground-state energy level of the anharmonic oscillator is also computed for arbitrary coupling strengths with good precision.

PACS numbers: 02.30.Hq, 02.30.Mv, 02.60.Lj, 11.27.+d

It occurs quite often in physics (but of course also in other areas of science) that we have to solve (singular) two-point boundary value problems (2p BVP) associated with a system of ODEs. An important class of such BVPs for linear ODEs arises from eigenvalue problems of the stationary Schrödinger equation either in one dimension or reduced to an ODE with some (e.g. rotational) symmetry. Another large class of 2p BVPs for *nonlinear* ODEs stems from the equations of motion of classical field theories reduced to ODEs (e.g. with some symmetries) and one could easily continue the list. We start with the example of the static, rotationally symmetric global vortex in a Ginzburg–Landau effective theory [1], which has numerous applications ranging from condensed matter to cosmic strings, see [2, 3]. The field equation determining the vortex profile can be written as

$$f''(r) + \frac{1}{r} f'(r) + \left(1 - \frac{n^2}{r^2}\right) f(r) - f^3(r) = 0, \tag{1}$$

where $f(r)$ is a real function, $f' = \mathrm{d}f/\mathrm{d}r$, and $n \in \mathbb{Z}$ corresponds to the vorticity. The physically interesting, globally regular solutions of equation (1) satisfy the following boundary conditions (BC):

$$f(r \to 0) = k_n r^n + \mathcal{O}(r^{n+2}), \qquad f(r \to \infty) = f_\infty = 1, \tag{2}$$

and then a major problem of the 2p. BVP amounts to find the value(s) of the free parameter, k_n, to ensure the BC of $f(r)$ at $r = \infty$.

It is not too difficult to integrate numerically equation (1), e.g., by the 'shooting' method from $r = 0$ to some large value of r and determine k_n to some precision but it is considerably more difficult to obtain analytical results. The aim of this communication is to present a new analytic procedure to approximate the value of k_n for the connecting trajectory involving only *algebraic* steps. Its basic input is the power series expansion of the solution around a point of analyticity (typically at $r = 0$) and the BC at $r = \infty$. With this input our method reduces the connection problem for k_n to find the corresponding root of a polynomial equation. The method is conceptually very simple, it is easy to apply and moreover it yields good approximations for various ODEs. Our method is heuristic, we cannot put it on a mathematically rigorous footing as yet, nor can we precisely define the class of ODEs to which it is applicable. Nevertheless, it seems to us that with regard to its simplicity and its large applicability the new method is of considerable interest for many applications (it yields with little effort good results for the connection parameters of many nonlinear ODEs, the energy levels of the quartic anharmonic oscillator for arbitrary values of the coupling, etc).

We illustrate our method in detail on the example of the global vortex (1). The first step is to introduce the following sequence of auxiliary first-order polynomial (implicit) ODEs of the form

$$F^N(f', f) := f'^N + G_1(f) f'^{N-1} + \cdots + G_{N-1}(f) f' + G_N(f) = 0, \qquad N = 1, 2 \ldots, \tag{3}$$

where $G_i(f) = \sum_j G_{ij} f^j$ is a polynomial in f with constant coefficients. Such implicit ODEs are not easy to handle in general, however, as it will be shown here, one can squeeze out some important information from the sequence $\{F^N(f', f) = 0\}$ without having to solve them. The next step in our method is the determination of the unknown coefficients G_{ij} in $F^N(f', f)$. In order to do this we enforce that equation (3) be satisfied to the highest possible order in r using the power series expansion of the solution of equation (1) $f(r)$, around $r = 0$. Having found the constants G_{ij} this way, we can now impose the BC at $r = \infty$ for $f(r)$ in equation (3) which amounts to

$$\sum_j G_{Nj} (f_\infty)^j = 0. \tag{4}$$

Since the coefficients G_{ij} depend on k_n (they turn out to be rational functions), equation (4) represents a polynomial equation for k_n. There is no *a priori* condition on the degree of the polynomials $G_i(f)$, we have chosen to impose $\deg(G_i) \leqslant 2i$. We remark that this restriction on the degree is known to be a necessary (but not sufficient) condition for the absence of movable branch points in equation (3). Our main observation is that in the set of real roots,

Table 1. Convergence of the approximants for the connection parameters of the global vortex for $n = 1, 2, 3, 4$.

N	k_1	k_2	k_3	k_4
3	0.585	–	–	–
4	0.5831	–	0.021	–
5	0.583 15	0.1527	0.025	–
6	0.583 190	0.1529	0.0264	0.0028
7	0.583 1894	0.153 10	0.026 183	0.0034
8	0.583 189 4936	0.153 09	0.026 185	0.0033
$k_{\rm num}$	0.583 189 4959	0.153 099 1029	0.026 183 420 72	0.003 327 173 40

$\mathcal{S}_N$, of (4) one can find a root, r_N, which seems to converge to the value of the connection parameter and the corresponding trajectory, $f_N(r)$, of (3) yields a global approximation to the solution of the 2p. BVP of equation (1). The main problem in our method is to identify the 'good' root in $\mathcal{S}_N$. At this point, we also note that our method performs a kind of resummation from the local power series expansion through the auxiliary ODEs, nevertheless we see no obvious relation to more standard resummation techniques such as the Borel technique or Padé approximants, see the monograph [4] for a nice review of these and other approximation techniques.

We now show in detail how our method works for the simplest case $N = 1$ and for vorticity $n = 1$. Without loosing generality one can assume $k_1 > 0$. In order to obtain a nontrivial result for $N = 1$ we have to take in equation (3) for $G_1(f)$ a polynomial of degree two, i.e. the simplest auxiliary ODE is a Ricatti equation:

$$f' + G_{10} + G_{11}f + G_{12}f^2 = 0. \tag{5}$$

Using the power series expansion of $f(r)$ (by solving equation (1)), $f(r) = k_1 r - k_1 r^3/8 + \cdots$, it is easy to obtain the coefficients G_{1i} in equation (5): $G_{10} = -k_1$, $G_{11} = 0$, $G_{12} = 3/(8k_1)$. Therefore, the solution of equation (4) corresponding to the BC (2) is $k_1 = \sqrt{3/8} = 0.612\ldots$, which constitutes a reasonable first approximation for k_1 (see table 1 for the numerical value, $k_{1\rm num}$) and by solving equation (5) one obtains quite a good approximation for the vortex profile function, $f(r)$.

To improve upon this approximation one could keep $N = 1$ fixed and increase only the degree of $G_1(f)$; however, since this scheme is not sufficiently general (it works quite well in some, but *not in all* cases), we rather consider the next member, $N = 2$, in our auxiliary ODE sequence with $\deg(G_i) = i$. In this case, we have to expand $f(r)$ in equation (1) up to order 5, and repeating the same procedure as for $N = 1$, one finds for the coefficients G_{ij}: $G_{10} = k_1\big(80k_1^2 + 1\big)\big/D_1$, $G_{11} = 0$, $G_{20} = -2k_1^2\big(20k_1^2 + 7\big)\big/D_1$, $G_{21} = 0$, $G_{22} = 81/(8D_1)$, with $D_1 = 13 - 40k_1^2$. Therefore, equation (4) reduces to

$$320k_1^4 + 112k_1^2 - 81 = 0, \tag{6}$$

whose positive real root is $k_1 = \sqrt{(-7 + \sqrt{454})/40} = 0.598\ldots$, which compares rather satisfactorily with $k_{1\rm num}$, considering the simplicity of the calculations. At this stage, it is natural to ask how this approximation changes keeping $N = 2$ fixed but increasing the degree of G_j to $2j$. Repeating the computations for this equation (4) yields $1 + 368k_1^2 - 2400k_1^4 + 3840k_1^6 = 0$, which has two real positive roots, $0.587\ldots$ and $0.531\ldots$. Now $0.587\ldots$, is a better approximation than the previous one, however there is also a second 'spurious' root, $0.531\ldots$, which needs to be excluded. The appearance of such 'spurious' roots for increasing N and for increasing degrees of G_j is a general feature and it is a main drawback of the method. In

Table 2. Convergence for the ground-state energy of the anharmonic oscillator, E_N for $N = 3, 4, 5$ and 10

β	E_3	E_4	E_5	E_{10}	E_v
0.2	0.51	0.562	0.5598	0.559 1455	0.559 1463
1	0.78	0.704	0.6984	0.696 1795	0.696 1758
2	0.90	0.813	0.8065	0.803 7773	0.803 7707
4	1.08	0.965	0.9548	0.951 5767	0.951 5685
100	2.85	2.56	2.502	2.498 3125	2.499 7088
400	4.49	4.02	3.931	3.930 989	3.930 9313
2000	7.65	6.86	6.692	6.694 321	6.694 2209
40 000	20.7	18.6	18.13	18.137 51	18.137 229
2×10^6	76.3	68.4	66.76	–	–
2×10^9	763	684	667.6	–	–

practice, however, this does not necessarily causes too serious problems, since one can easily follow the 'good' roots by continuity and checking their stability against changing N and the degrees of G_j. In the following, unless indicated otherwise, we shall choose the degree of the polynomials G_j to be j, this fixes the *global degree* of $F^N(f', f)$ to be N. In this case, there are altogether $(N+1)(N+2)/2$ constants G_{ij} to be determined, and the order of the power series in r to be used has to be chosen accordingly. We present the approximate values of the connection parameter k_n for $n = 1, \ldots, 4$ up to $N = 8$ in table 1. Remarkably for $n = 1$ the $N = 8$ approximation yields eight correct digits compared with $k_{1\mathrm{num}}$. For increasing values of n the order of the series in r must also be increased for a given degree of $F^N(f', f)$ and consequently the degree of the polynomial in k_n is greater. We stress that all computations are analytic (they have been performed on a standard desktop computer using Mathematica 5.2), except to find the roots of the corresponding polynomial where numerical methods became inevitable.

Next we show that our method also yields good results for an important eigenvalue problem, the determination of the energy levels of the quartic anharmonic oscillator in 1 dimension. Using dimensionless variables, the Schrödinger equation can be written as

$$f''(x) + (2E - \beta x^4 - x^2) f(x) = 0, \tag{7}$$

where β is the coupling parameter and E is the energy eigenvalue[3]. The 2p. BPV for the ground-state wavefunction (which is even) can be put in the form $f(x=0) = 1$, $f(x \to \infty) = 0$. For $N = 3$, one obtains the following algebraic equation for the eigenvalue E:

$$-6408E^6 + 12960\beta E^5 + 2356E^4 - 2976\beta E^3 - 2(1440\beta^2 + 133)E^2 + 168\beta E + 25 = 0. \tag{8}$$

The harmonic oscillator corresponds to $\beta = 0$, whose exact ground-state energy is $E = 1/2$. For $\beta = 0$ there is a single positive real root of equation (8), $0.5166\ldots$, and the choice of the roots for $N = 3$, E_3, in table 2 has been done by following this root as β has been varied. We summarize our results for $N = 3, 4, 5$and10 in table 2, where we also compare them with the known ones in the literature [5] denoted as E_v. One can thus see that our method yields very good approximate values for the ground-state energy of the anharmonic oscillator for *all values* of the coupling. We remark that some of the other real roots of the polynomial in E are related to the energy levels of the excited states and by our method one can also obtain approximate values for them, but we will not elaborate on this point here.

[3] Note that our β corresponds to $g/2$ of [5].

Let us next present here some results on the simplest 'skyrmion' solution, which is of considerable interest as a good approximation for baryons, we refer to the recent monograph [3] for details. The pertinent ODE for the spherically symmetric skyrmion field can be written as

$$(r^2 f')' + 2f'' \sin^2 f + \sin(2f)[f'^2 - 1 - (\sin^2 f)/r^2] = 0, \tag{9}$$

together with the BCs $f(r) = \pi + kr + \mathcal{O}(r^3)$, $f \to 0$ for $r \to \infty$. In this case when applying our method we have found that the choice $\deg(G_i(f)) = 2i$ yields significantly better results than $\deg(G_i(f)) = i$. This way we find for the connection parameter: $k = -2.084$ for $N = 2$; $k = -1.996$ for $N = 3$ and $= -2.003$ for $N = 4$, whereas $k_{\mathrm{num}} = -2.007$. The agreement is quite satisfactory taking into account the relatively low degree of the auxiliary equations (3).

An important physical quantity is the total energy of such localized solutions. For example, the energy of the skyrmion in a ball of radius R is given as $E(R) = \int_0^R \mathcal{E}\,\mathrm{d}r$, where $\mathcal{E}$ is the energy density:

$$\mathcal{E} = [r^2 f'^2 + 2(1 + f'^2)\sin^2 f + (\sin^4 f)/r^2]/(3\pi), \tag{10}$$

and the *total energy* is then $E(\infty)$. The direct way to compute approximate values for the energy, i.e., finding the corresponding solutions of the auxiliary ODE (3) first and then evaluating $E(R)$ for them would be quite difficult without resorting to numerics. We can apply a slight variation of our method, however, to obtain approximate values for the total energy once the value of the connection parameter is known. To do this, we consider the sequence $\{F^N(\mathcal{E}, E) = 0\}$. Since the power series expansion of the energy, $E(r)$, is completely fixed for any given value of the connection parameter, k, all the coefficients, G_{ij}, will be determined. Then, we solve the corresponding algebraic equation (4) for the unknown value of $E(\infty)$. This way for $N = 9$ we have obtained $E(\infty) = 1.22$, which seems to be quite good when compared to the numerical value, $E_{\mathrm{num}} = 1.23$.

We consider now some ODEs originating from a completely different problem, the fixed point equation of Wilson's exact renormalization group (RG). The RG equation for scalar field theories in the local potential approximation can be written as [6]

$$2f''(x) - 4f(x)f'(x) - 5xf'(x) + f(x) = 0, \tag{11}$$

where $f(x) = V'(x) - x$ and $V(x)$ is the potential. The pertinent solution of (11) is an odd function of x and for $x \to 0$ $f(x) = kx + \mathcal{O}(x^3)$. For large values of x $f(x \to \infty) \to ax^{1/5}$ where a is a constant. Since in the present case $f_\infty = \infty$, $f'' \to 0$ and $f' \to 0$ for $x \to \infty$, it is rather natural to slightly modify the method by considering the auxiliary ODEs (3) for (f'', f'), i.e. $F^N(f'', f') = 0$. Then proceeding exactly as before we obtain a polynomial equation in k, and we find with a good convergence $k = -1.228\,598\,76$ for $N = 6$. This agrees quite well with the numerical value $k_{\mathrm{num}} = -1.228\,598\,20$ [7].

Let us consider yet another example, the Wegner–Houghton's fixed point equation in the local potential approximation [8],

$$2\ln(1 + v''(x)) + 6v(x) - xv'(x) = 0. \tag{12}$$

The change of dependent variable, $f(x) = v'(x)$, gives the simpler differential equation

$$2f''(x) + [1 + f'(x)][5f(x) - xf'(x)] = 0. \tag{13}$$

The solution of interest satisfies the following BCs: $f(x \to 0) = kx + \mathcal{O}(x^3)$ and $f(x \to \infty) \to ax^5$ where a is a constant. The connection parameter, k, obtained numerically is $k_{\mathrm{num}} = -0.461\,533$ [6]. The problem is similar to the precedent one but we use now higher derivatives, $F^N(f^{(7)}, f^{(6)}) = 0$. Proceeding in the same way that in the previous case we

Table 3. Connection parameters for the vortex and the magnetic monopole with $N_1 = 3$, $N_2 = 4$.

β	c_1	c_2	$c_{1\text{num}}$	$c_{2\text{num}}$	d_1	d_2	$d_{1\text{num}}$	$d_{2\text{num}}$
0	–	–	–	–	0.3329	0.1523	1/3	1/6
1	0.8542	0.5007	0.8532	0.5000	0.7343	0.3422	0.7318	0.3409
2	1.0996	0.6169	1.0993	0.6166	1.0692	0.4501	1.0683	0.4491
3	1.2846	0.6975	1.2843	0.6969	1.4025	0.5350	1.4003	0.5321
4	1.4393	0.7609	1.4387	0.7597	1.7406	0.6100	1.7339	0.5997
5	1.5748	0.8137	1.5741	0.8119	2.0872	0.6901	2.0701	0.6567
6	1.6972	0.8595	1.6962	0.8569	2.4421	0.7834	2.4091	0.7060
7	1.8097	0.9001	1.8082	0.8967	2.8035	0.8938	2.7503	0.7492
10	2.1052	1.0011	2.1024	0.9944	3.6269	0.9678	3.7850	0.8542

obtain $k = -0.461\,44\ldots$ for $N = 6$ which agrees with the numerical value, although not so well as in the previous case, probably due to the use of derivatives of higher order.

We now show that our method can be generalized in a very simple way for a system of M ODEs for the set of unknowns $\{f_m(r)\}$, $m = 1, \ldots, M$. We introduce for each unknown function a sequence of first-order auxiliary implicit ODE's $\left\{F_m^{N_m}(f_m', f_m) = 0\right\}$, where $F_m^{N_m}(f_m', f_m)$ are polynomials of degree N_m in f_m' (cf equation (3)). The constant coefficients in $F_m^{N_m}(f_m', f_m)$ are determined by demanding that $\left\{F_m^{N_m}(f_m', f_m) = 0\right\}$ be satisfied to the highest possible order in the power series solutions of f_m at the origin, say. Then proceeding exactly as for the case of a single unknown we impose the BC at infinity and obtain a system of algebraic equations of the form $\sum_j G_{N_m j}\left(f_m^j(\infty)\right) = 0$ for the connection parameters. As a concrete illustration we shall consider the field equations of the static, rotationally symmetric, gauged vortex of Nielsen–Olesen [9] and those of the 't Hooft–Polyakov magnetic monopole [10].

The differential equations determining the cylindrically symmetric magnetic potential resp. scalar fields, a, f in the plane of the Nielsen–Olesen vortex with a single magnetic flux quantum can be written as

$$r(rf')' - f[(1-a)^2 - r^2\beta(1-f^2)] = 0, \tag{14a}$$

$$ra'' - a' + 2r(1-a)f^2 = 0, \tag{14b}$$

where β corresponds to the self-coupling of the scalar field. The BCs necessary to ensure regularity and finite energy at $r = 0$ and at $r = \infty$ are: $f = f_1 = c_1 r + \mathcal{O}(r^3)$, $a = f_2 = c_2 r^2 + \mathcal{O}(r^4)$, $f(\infty) = 1, a(\infty) = 1$. To implement our method we have chosen $F_1^{N_1}(f_1', f_1)$ to be a polynomial of degree N_1 and $F_2^{N_2}(f_2', f_2)$ to be a polynomial of degree $N_2 = N_1 + 1$ in order to have roughly the same number of terms in the algebraic equations for c_1, c_2. Next we recall the differential equations for the static spherically symmetric magnetic monopoles in a spontaneously broken $SU(2)$ gauge theory:

$$(r^2 f_1')' - f_1\left[2f_2^2 + \frac{r^2}{2}\beta^2\left(f_1^2 - 1\right)\right] = 0, \tag{15a}$$

$$r^2 f_2'' - f_2\left[\left(f_2^2 - 1\right) + r^2 f_1^2\right] = 0, \tag{15b}$$

where f_1 and f_2 correspond to the Higgs and the gauge field, respectively. The regular BCs at $r = 0$ resp. $r = \infty$ are $f_1 = d_1 r + \mathcal{O}(r^3)$, $f_2 = 1 - d_2 r^2 + \mathcal{O}(r^4)$, $f_1(\infty) = 1$ and $f_2(\infty) = 0$. We present in table 3 our results for the connection parameters of both the vortex and the magnetic monopole for $N_1 = 3$ and $N_2 = 4$, and the corresponding numerical values from [11, 12]. The agreement is good, taking into account that the computations involving two polynomials are more cumbersome; moreover, for both for small and large values of β, one needs to increase the values of N_1 and N_2 more and more. We recall here that for $\beta = 0$ equations (15*a*) are analytically soluble, yielding $d_1 = 1/3$, $d_2 = 1/6$. We have also computed the total energy for $\beta = 1$ applying the same procedure as for the skyrmion, and we have obtained $E(\infty) = 1.2136$ for $N = 10$ with $\deg(G_j) = 2j$ (cf $E_{\mathrm{num}} = 1.237$).

In conclusion, we have presented a new algebraic scheme to obtain the connection parameters, the energy eigenvalues and the total energy for a number of physically interesting two-point boundary value problems associated with ODEs. The method is based on a sequence of auxiliary first-order implicit polynomial ODEs, which is determined from the series expansion of the solution to a given degree. Imposing the BCs for the solution of the auxiliary ODEs yields algebraic equations for the connection parameters. The computation of power series expansions and finding the roots of the algebraic equations are easily implementable on symbolic formula manipulation systems. It seems to be an interesting problem to clarify the mathematical basis of our method.

Acknowledgments

We would like thank Claude Bervillier for pointing out that the fixed point equations of the RG serve as interesting illustrations of our method and for his interest. We also thank him and Peter Breitenlohner for providing us various numerical results. One of us (BB) had also some interesting discussions with Stam Nicolis.

References

[1] Ginzburg V L and Pitaevskii L P 1958 *Sov. Phys.—JETP* **34** 858
Pitaevskii L P 1961 *Sov. Phys.—JETP* **13** 451
[2] Vilenkin A and Shellard E P S 1994 *Cosmic Strings and Other Topological Defects* (Cambridge: Cambridge University Press)
[3] Manton N and Sutcliffe P 2004 *Topological Solitons* (Cambridge: Cambridge University Press)
[4] Bender C M and Ország S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)
[5] Karrlein R and Kleinert H 1994 *Phys. Lett.* A **187** 133
[6] Bagnuls C and Bervillier C 2001 *Phys Rep.* **348** 91
[7] Bervillier C 2006 Private communication
[8] Hasenfratz A and Hasenfratz P 1986 *Nucl. Phys.* B **270** 687
[9] Nielsen H B and Olesen P 1973 *Nucl. Phys.* B **61** 45
[10] 't Hooft G 1974 *Nucl. Phys.* B **79** 276
Polyakov A M 1974 *JETP Lett.* **20** 194
[11] Forgács P, Reuillon S and Volkov M S 2006 *Nucl. Phys.* B **751** 390
[12] Forgács P, Obadia N and Reuillon S 2005 *Phys. Rev.* D **71** 035002–1